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Bénard–von Kármán instability: transient and forced regimes

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The wake of a circular cylinder is investigated near the oscillation threshold by means of a laser probe. Above the threshold the transient regime is studied and described by a Stuart–Landau law (already found to be relevant in explaining free-oscillating regimes). Below the critical point, impulse and resonant regimes are examined, so the coefficients of the Stuart–Landau equation are determined.

Moreover, in the supercritical regime, the behaviour of the (externally forced) oscillating system is described, varying parameters such as threshold deviation, forcing frequency and amplitude. The different zones of entrainment and desynchronization are given for simple or harmonic frequency.

1. Introduction

In spite of the vast amount of literature dealing with the problem of the wake behind a cylinder, interest in this problem has really increased. There are three reasons for this.

The first is associated with the development of numerical simulations which now are able to handle three-dimensional problems on some configurations (Gollub & Freilich 1981; Libchaber & Maurer 1981; Haldenwang & Labrosse 1986; MacLaughlin & Orszag 1982) but not yet for the cylinder's wake to our knowledge. The second is related to the use of non-intrusive laser diagnostics which give local and instantaneous measurements and therefore new information. The third, the most important in our opinion, is of a conceptual nature: the renewed interest in dynamical systems with their application to a large class of nonlinear phenomena such as phase transitions and instabilities in various domains. In the last ten years, the transition to turbulence has been studied extensively and reported through excellent works concerning internal flows (Rayleigh–Bénard and Taylor instabilities) but, perhaps due to their complexity, the external flows have been less investigated. Now, the practical importance, diversity and specific properties (spatial characteristics) of these flows are such that it seems worthwhile to work on them as well. With these considerations in mind, we have investigated the oscillation of the wake downstream from a cylinder. This breakdown of symmetry is generally considered as a first step in the route to chaos (Joseph 1981).

The initial focus concerns the nature of the instability and the adequacy of a model. Moreover the coexistence of two oscillating modes in a narrow range of Reynolds number, which has been discussed in a long debate without any conclusion (Tritton 1971; Gaster 1971; Berger & Wille 1972), is confirmed. On the other hand it was interesting to study the succession of instabilities with a view to characterizing the transition to chaos, following the ideas of either Landau–Lifshitz (1971) or Ruelle–Takens (1979). Eventually, given a model, it was of interest to check its properties from an experimental point of view and to define its domain of applicability.

Let us summarize briefly the different regimes as functions of the Reynolds number (generally defined with the axial mean velocity $U_{x\infty}$ far upstream, the diameter d and the kinematic viscosity ν). For very small Reynolds number ($Re \ll 1$) the flow is creeping and the viscosity terms are preponderant. For higher Reynolds number, the flow is still steady but there is a recirculation zone just behind the cylinder which consists of two symmetrical and fixed eddies increasing with Re (Coutanceau & Bouard 1977). The existence of the discontinuity in the evolution of the flow (i.e. the emergence of the recirculating zone) is commonly admitted, but the work of Nakamura (1976) on the sphere and the range of the threshold value ($2 < Re < 6$) lead us to think that the recirculation zone is always present. A fine study of this problem seems desirable. Up to the critical value Re_c, the strength of the eddies increases and the zone stretches downstream; at the threshold the oscillation of the wake begins. It is associated with the formation of a double row of opposing vortices convected downstream. This interpretation was given by Henri Bénard in 1908, while von Kármán (1911) performed the first stability analysis of the structure without reference to the mode of formation, which is not yet clearly determined to our knowledge (Tritton 1971). Nevertheless we do mention the interesting numerical simulations which describe correctly the mean features of the flow (Fornberg 1980), the dynamics of the unsteady flow (Braza 1981) and the stability analysis of the fully-developed vortex street (Meiron, Saffman & Schatzman 1984; Saffman & Schatzman 1982).

It can be noticed simply that for Re_c there is a real transition characterized by a loss of symmetry and non-stationarity of the flow. This transition is the principal focus of this work. The Reynolds-number range is restricted to $40 < Re < 300$ (in our experimental conditions Re_c varies with the cylinder's aspect ratio from 47 to 123). The ignorance on the nature of the transition from stationary to oscillating regime and the lack of linear stability analysis of this flow required an accurate investigation in this range. It has been done using laser Doppler anemometry with a moving-fringe system well-adapted to the study. Information is lacking on several points:

The oscillation threshold is still not well known.
There is no theory concerning the formation modes; the existence or the non-existence of three-dimensional structures is still being discussed.
Do the oscillations start with zero amplitude or is there any hysteresis?
What is the growth law as a function of the deviation $Re–Re_c$?
What is the route to turbulence?
Is any intermittency possible near the threshold?

The first four points have been developed. The experimental conditions have led us to evaluate the role of the confinement on different parameters such as the critical Reynolds number at the threshold. The three-dimensional effects (birth of two

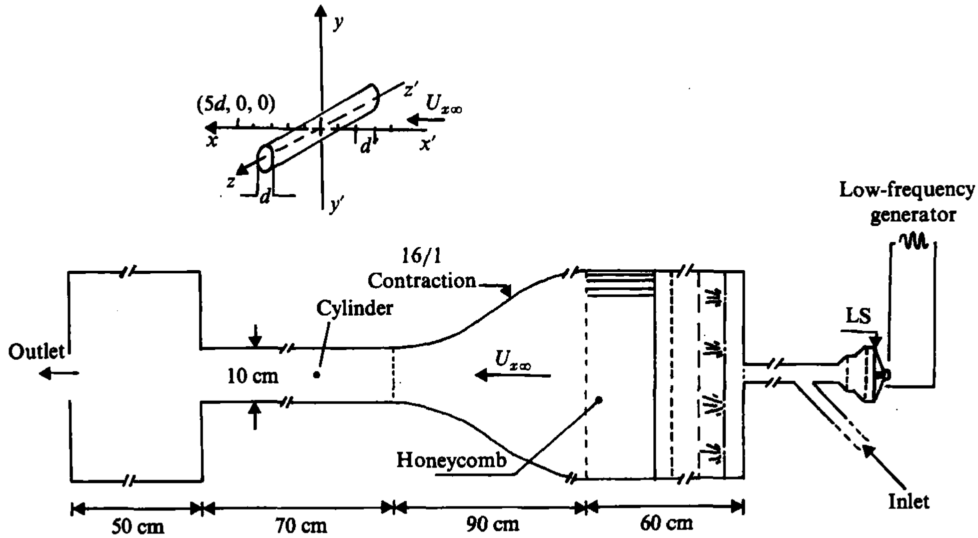

FIGURE 1. The wind tunnel with the excitating device. The origin is on the axis of the cylinder.

oscillatory modes) have been observed and associated with the shedding of inclined vortex filaments.

One of the main results concerns the evolution of the order parameter of the bifurcation, the transverse oscillating velocity. This law has been determined with accuracy, as well as the behaviour of the system in transient regimes.

The first elements of this study have been published (Mathis, Provansal & Boyer 1984*a*, *b*) but a short summary is given. These previous results mainly show that the transition is characterized by a Hopf bifurcation and can be described by a Stuart–Landau law. This is the first evidence of such behaviour for external flows to our knowledge. In the present paper we focus our attention on the transient dynamical properties of the system as well as on its response in forced regimes below and beyond the threshold.

See Appendix for notation.

2. Experimental apparatus

The experiments were conducted in air, downstream $(5d, 0, 0)$ from a circular cylinder located in a square element (cross-sectional width $L = 10$ cm), of an open-circuit wind tunnel characterized by a 16/1 contraction ratio (figure 1). A sonic nozzle is used for regulating the flow; the correction of thermal effects and the smoothness of steel cylinders allow an accuracy of better than 0.5% on the determination of the Reynolds number. The upstream flow has a flat profile (1.5%) and the free-stream turbulence level is less than 0.3%.

The sensitivity of the system to perturbations requires the choice of a non-intrusive diagnostic, laser Doppler anemometry, which is a well-adapted technique. Moreover, a rotating grating, both splitting the beam and moving fringes, gives the algebraic value of the components of the flow and increases the accuracy of measurements (Mathis *et al.* 1984*b*).

A TSI signal processor and a Rockland FFT 512S or Solartron 1200 spectrum analyser were used. In forced and impulsed experiments, a loudspeaker (LS) or a moving diaphragm made of rubber were used, positioned on the axis upstream of the wind tunnel by way of a Y channel (figure 1).

3. Study of the amplitude in the transient and the impulse regimes

3.1. *The Landau model*

In order to throw some light on the discussion, we propose to recall briefly the place of the Landau model in hydrodynamics. The approach starts from the attempt to perform the stability analysis of a steady flow. A non-stationary perturbation $u_1(x,y,z,t)$ of the steady solution $u_0(x,y,z)$ of the Navier–Stokes equations is expanded as a sum:

$$u_1(x,y,z,t) = \sum_{i=1} \{A_i(t)\,g_i(x,y,z) + A_i^*(t)\,g_i(x,y,z)\},$$

where $g_i(x,y,z)$ satisfies the boundary conditions. Then it can be shown that the amplitude $A_i(t)$ satisfies the evolution equation

$$\frac{\mathrm{d}A_i}{\mathrm{d}t} = s_i A_i + G_i(A_j) \quad (j = 1, 2, \ldots),$$

where G_i contains the nonlinear interaction of all the modes (mode i included) resulting from the nonlinear partial differential equation. The Landau equation is then a truncated form of the above equation.

Let us consider an independent mode $A(t) \propto \mathrm{e}^{\sigma t}$ with relative growth rate $\sigma = \sigma_r + \mathrm{i}\sigma_i$.

When $Re < Re_c$, all disturbances are stable and $\sigma_r < 0$.

When $Re = Re_c$ there is just one normal mode with $\sigma_1 = \sigma_{r1} + \mathrm{i}\sigma_{i1} (\sigma_{r1} = 0$ for $Re = Re_c)$ which is marginally stable.

As Re increases above Re_c, $\sigma_{r1} > 0$ but $\sigma_r < 0$ for all the other modes. The expression for the amplitude $A \propto \mathrm{e}^{\sigma t}$ is no longer valid for long times; obviously the modulus does not grow infinitely but is bounded. The Landau hypothesis leads us to search for this limit: for short time $|A(t)| \propto \mathrm{e}^{\sigma_r t}$ is valid, so that

$$\frac{\mathrm{d}|A(t)|^2}{\mathrm{d}t} = 2\sigma_r |A(t)|^2. \tag{1}$$

This expression is the first term of the expansion of the solution in powers of A and A^*. The second term is of third order in $|A|$. However, we are not interested in $\mathrm{d}|A|^2/\mathrm{d}t$ but in the average on times τ long compared to the period $T = 2\pi/\sigma_1$ but small enough to keep u_1 as a small perturbation. These two constraints define the valid area of the Landau equation; they are compatible if $\sigma_r \ll \sigma_i$, which is true near the threshold. The mean value of the third-order term is null (in fact not exactly null but giving a fourth-order contribution), so that the second term is of fourth power:

$$\frac{\mathrm{d}|A|^2}{\mathrm{d}t} = 2\sigma_r |A|^2 - l_r |A|^4,$$

or

$$\frac{\mathrm{d}|A|}{\mathrm{d}t} = \sigma_r |A| - \tfrac{1}{2} l_r |A|^3, \tag{2}$$

which is the equation proposed by Landau in 1944.

The steady solution for the amplitude (of the limit cycle in the vocabulary of dynamical systems) is $|A| = (2\sigma_r/l_r)^{\frac{1}{2}}$. Moreover, it must be noticed that σ_r plays the role of a bifurcation parameter, which is equal to zero at the threshold. When the instability starts, we can develop

$$\sigma_r = k(Re - Re_c) + O(Re - Re_c)^2, \tag{3}$$

where k is some positive constant (characteristic frequency $\nu/5d^2$ in our problem) and we obtain for the amplitude:

$$|A| \,\alpha (Re - Re_c)^{\frac{1}{2}}. \tag{4}$$

This behaviour is generally compared to a second-order phase transition. Landau did not show how his equation could be derived for the stability of a given flow. This work has been done by Stuart (1958, 1960) for plane parallel flows and Palm (1960) for the Rayleigh–Bénard problem. Further references can be found in Drazin & Reid (1981) or in Kuramoto (1984). Following Kuramoto we will call Stuart–Landau the equation:

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \sigma A - \tfrac{1}{2}l|A|^2A \tag{5}$$

which gives in amplitude the Landau equation

$$\frac{\mathrm{d}|A|}{\mathrm{d}t} = \sigma_r|A| - \tfrac{1}{2}l_r|A|^3 \tag{2'}$$

and in phase

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \sigma_i - \tfrac{1}{2}l_i|A|^2 \tag{6}$$

3.2. *Summary of previous work*

The first fundamental results have been reported in the thesis of Mathis in 1983. They have been confirmed in a similar but independent work done by Strykowski in 1986:

The amplitude $|u_y|$ of oscillatory transverse velocity, taken as the order parameter, verifies a Landau's law (4) $|u_y| = \alpha(Re - Re_c)^{\frac{1}{2}}$, in which Re is the order parameter of the bifurcation and the exponent $\frac{1}{2}$ is known with an accuracy better than 1%.

The critical Reynolds number Re_c strongly varies with the aspect ratio L/d (measurements have been made by changing the length L or the diameter d). The asymptotic value $Re_c(L/d = \infty) = 47$ is pretty close to the value computed by Jackson (1987) who performed a numerical study of this Hopf bifurcation.

The cofactor α depends upon the aspect ratio and the measurement point in the wake.

The well-known relation Ro/Re (Roshko 1954) between the dimensionless frequency $Ro = fd^2/\nu$ and the Reynolds number, has been experimentally checked and found compatible with the Stuart–Landau model. However, for cylinders of large aspect ratio, there is a range of Reynolds numbers where two oscillatory modes both appear due to three-dimensional effects; measurements on $u_z(t)$ and visualizations by means of a laser sheet allow one to interpret this phenomenon as the consequence of a small initial speed gradient $\nabla U_{x\infty} \neq 0$ (such a phenomenon was also noticed by Gaster 1971; Sreenivasan 1985; Tritton 1971).

3.3. *Study of the amplitude in the transient regime*

The Stuart–Landau model has been checked through the time evolution of the amplitude. The Landau equation in amplitude is:

$$\frac{\mathrm{d}|u_y|}{\mathrm{d}t} = \sigma_r|u_y| - \tfrac{1}{2}l_r|u_y|^3, \tag{2''}$$

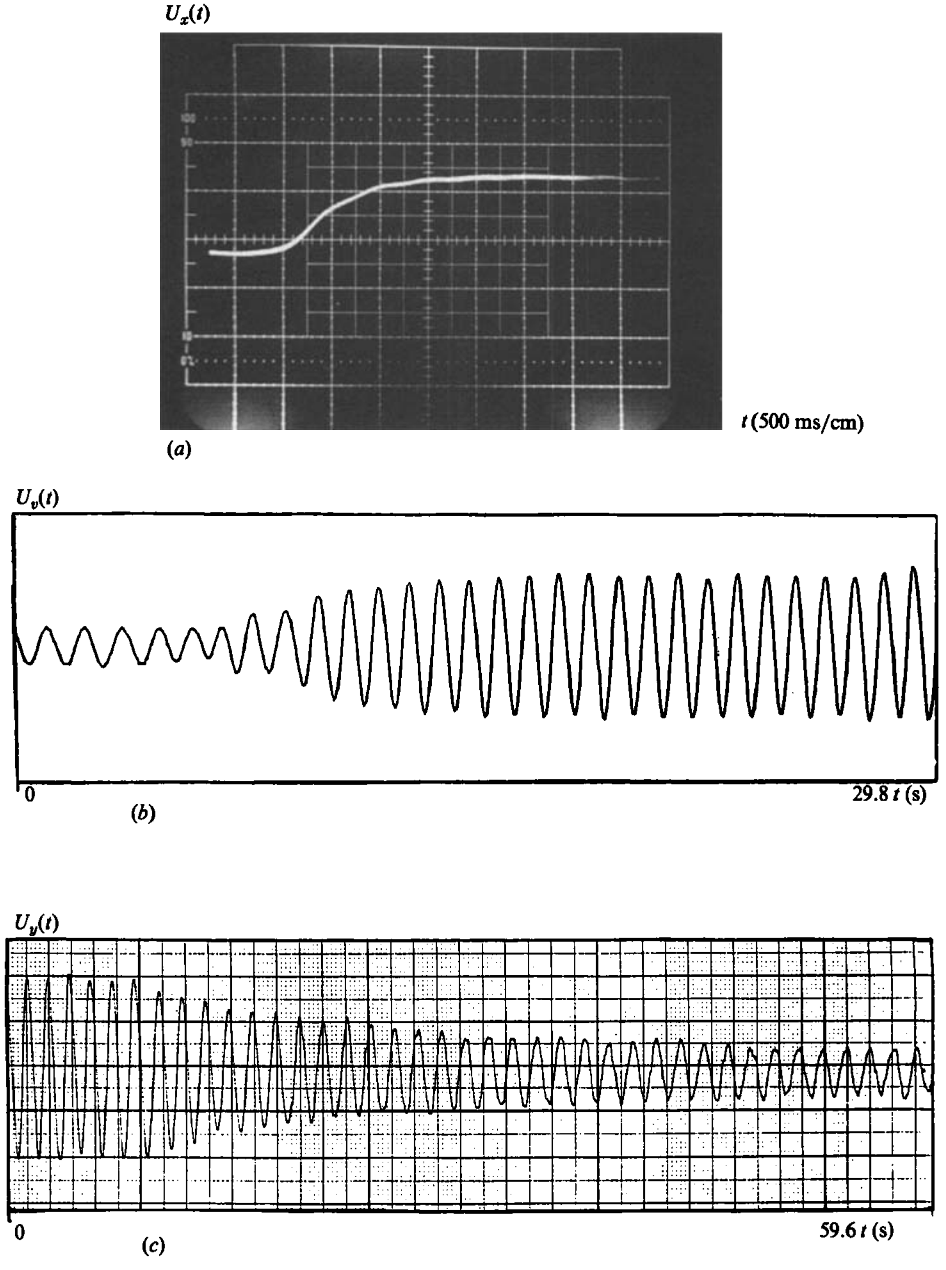

FIGURE 2. (*a*) $u_x(t)$ upstream of the cylinder in transient regimes: $Re_i = 107$, $Re_f = 125$. (*b*, *c*) $u_y(t)$ in transient regimes: typical recordings. (*b*) $d = 1.6$ cm, $Re_i = 107$, $Re_f = 125$, $Re_c = 102$; (*c*) $d = 2.0$ cm, $Re_i = 139$, $Re_f = 127$, $Re_c = 123$. Notice the inflexion point in the first case.

writing initial and final conditions, $|u_y|\,(t=0) = |u_{yi}|$ and $|u_y|\,(t=\infty) = |y_{yf}|$, we get the solution:

$$|u_y|\,(t) = |u_{yf}|\left[1+\left(\frac{|u_{yf}|^2}{|u_{yi}|^2}-1\right)\mathrm{e}^{-2\sigma_r t}\right]^{-\frac{1}{2}}. \tag{7}$$

The transient behaviour is characterized by the evolution term $\mathrm{e}^{-2\sigma_r t}$ (where $\sigma_r = k(Re-Re_c)$) and by the time $\tau = 1/\sigma_r$. Similar work was done by Gollub &

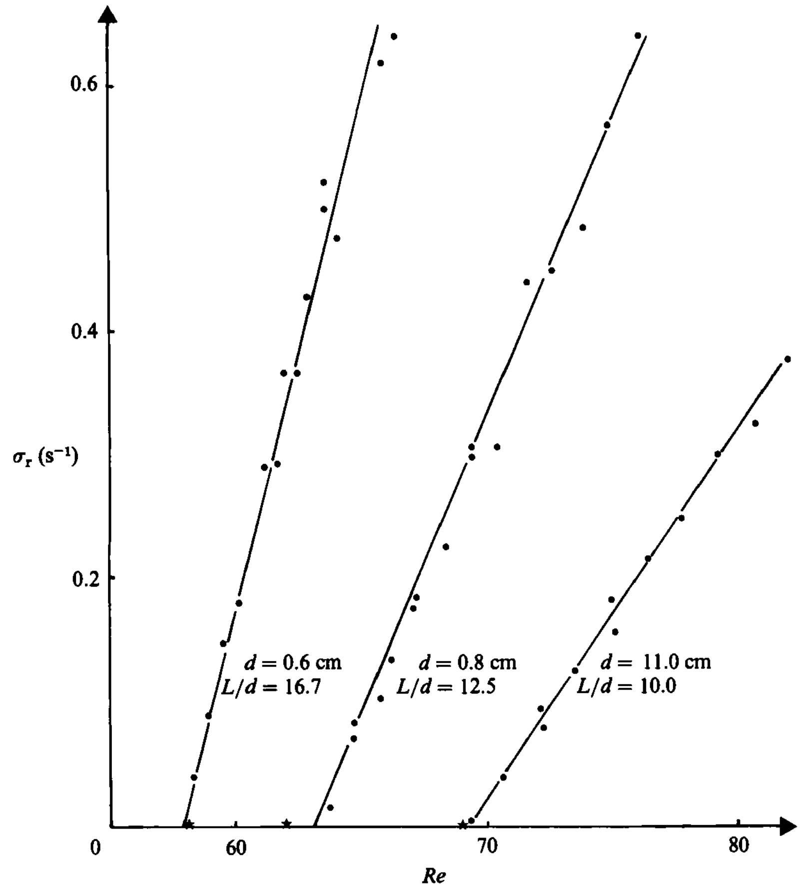

FIGURE 3. $\sigma_r(Re)$ with $d = 0.60$ cm, $d = 0.80$ cm, $d = 1.00$ cm. $L/d = 16.7$, $L/d = 12.5$, $L/d = 10$. *: Re_c as previously determined.

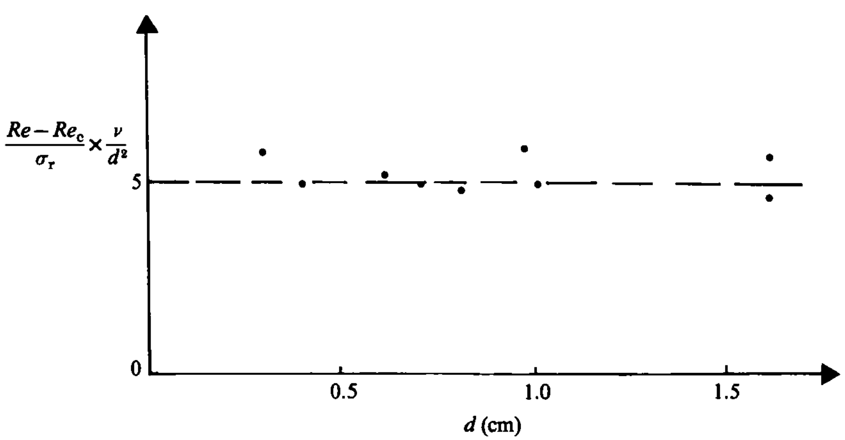

FIGURE 4. The dependence of σ_r with the diffusivity – time for various diameters.

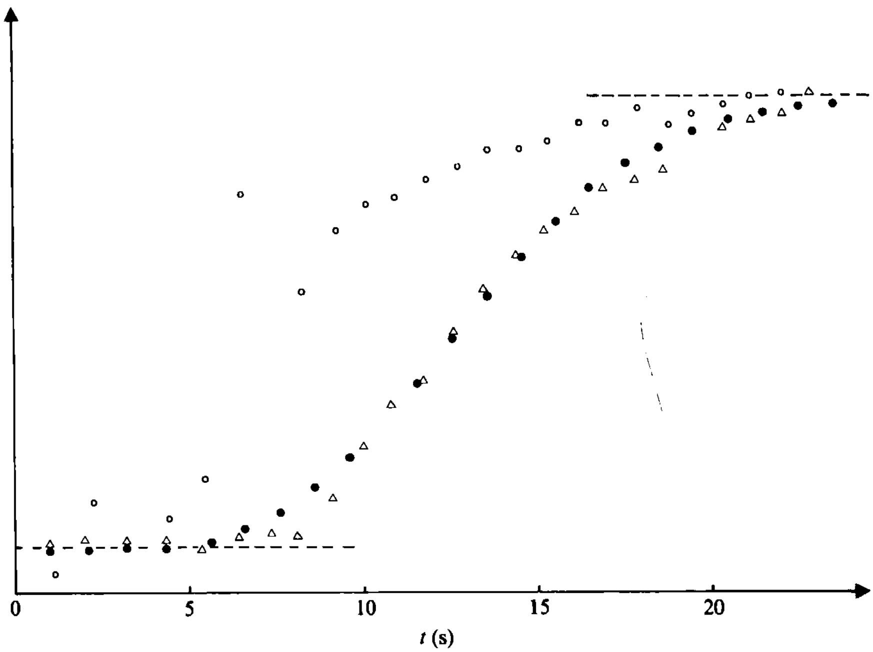

FIGURE 5. Simultaneous evolution of $|u_y|^2(t)$ and $f(t)$. △, $|u_y|^2{}_{\text{exp}}$ (arbitrary units) is deduced from the recordings. ○, frequency (Hz). f varies from 0.93 Hz to 1.18 Hz. ●, $|u_y|^2(t)_{\text{computed}}$ (arbitrary units) is the plot of equation (7) in which $|u_{yi}|^2$, $|u_{yf}|^2$ are experimental and σ_r is deduced from equation (8) and the values $d = 1.60$ cm, $Re_i - Re_c = 1.5$, $Re_f - Re_c = 22.6$, $|u_{yi}|^2 = 1.5$, $|u_{yf}|^2 = 22.6$ (arbitrary units).

Freilich (1981) concerning the Taylor instability and by Libchaber & Maurer (1981) who characterized the Landau behaviour of the first oscillation in the Rayleigh–Bénard problem.

Experimentally the Reynolds number is quickly changed, by modifying the upstream pressure in a time (0.5–1.0 s) of the same order as a period, from Re_i to Re_f (figure 2*a*) and we have determined τ by analysis of recorded graphs on $|u_y|\,(t)$ (figure 2*b, c*). Measurements have been done for eight cylinders whose diameters are in the range (0.30, 2.00) cm. These recordings fit the computed pattern (figure 5) in both the cases $Re_i > Re_f$ and $Re_i < Re_f$. In figure 3, the graphs of the variation $\sigma_r(Re)$ are given for three cylinders. The lower part of the graph gives the critical Reynolds number which is in good agreement with that deduced from the stationary study and relation (4): $|u_y|^2 = \alpha^2(Re - Re_c)$. The determination of the slopes of $\sigma_r(Re)$ allows one to write

$$\sigma_r = \frac{(Re - Re_c)\,\nu}{5d^2}, \tag{8}$$

where d^2/ν is the viscous diffusion time which is a characteristic of each cylinder, and has been incorporated into the Roshko number (figure 4). In the same manner (i.e. by quick variation of the upstream pressure), we could deduce the pseudo-period (i.e. the time between two successive null values of the amplitude) of the oscillation. The simultaneous evolution of the amplitude and of the frequency are plotted in figure 5.

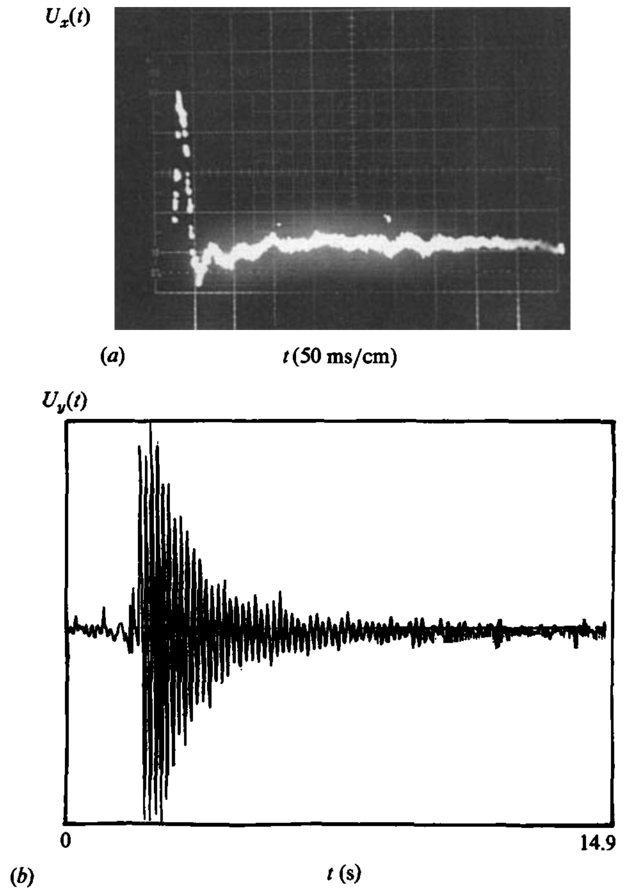

FIGURE 6. (a) Recording of the impulse on $u_x(t)$, upstream of the cylinder. (b) Typical recording of the impulse response on $u_y(t)$: $d = 0.4$ cm, $Re - Re_c = -1.4$.

3.4. *Impulse response*

Below the critical Reynolds number, the linear stability analysis is sufficient to describe the dynamics of a small perturbation in the stream around the cylinder. Assuming that the nonlinear term is negligible, the equation of evolution becomes:

$$\frac{du_y}{dt} = \sigma u_y, \tag{9}$$

where $\sigma = \sigma_r + i\sigma_i$ and the solution is

$$u_y(t) = |u_y| \, e^{\sigma_r t} \, e^{i\sigma_i t}. \tag{10}$$

To test this hypothesis and check the characteristic time evolution $\tau = 1/|\sigma_r|$ the system is perturbed by an impulse. Experimentally this is done by striking a thin elastic diaphragm located at the entrance of the wind tunnel instead of the loudspeaker (figure 6*a*, *b*). The variation of this relaxation time is plotted versus the Reynolds number (figure 7). The relaxation time is still expressed as

$$\tau = \frac{5d^2}{\nu|Re - Re_c|} = \frac{1}{|\sigma_r|} \tag{8'}$$

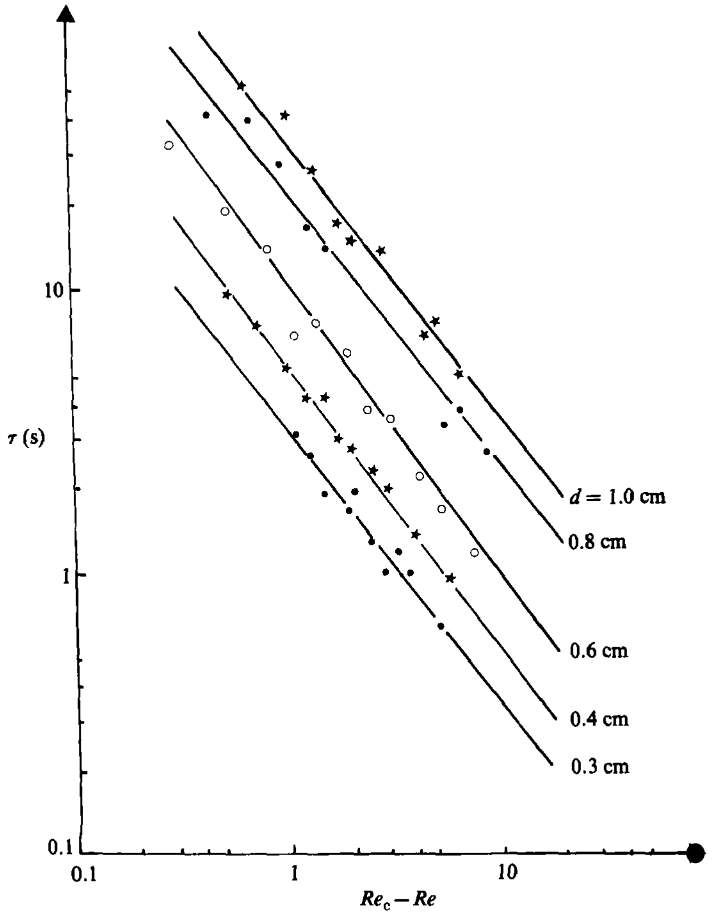

FIGURE 7. Relaxation time versus $Re - Re_c$ in a log–log plot.

with an accuracy better than 5%. This expression confirms the result reported above for a transient regime. It can be interesting to point out that not only is relation (8) confirmed by Strykowsky (1986) but the numerical value of the coefficient is the same, equal to 5. Thus in transient and impulse regimes for the subcritical range as well as for the supercritical area the time evolution of the amplitude verifies the Stuart–Landau model. Moreover the dimensionless frequency of the relaxed oscillation is given by the same relation as the natural frequency

$$Ro = aRe + b. \tag{11}$$

4. Forced oscillations

4.1. *Resonance in the subcritical regime*

From the relation Ro/Re we deduced by extrapolation the 'natural' frequency f_1 of the system when the Reynolds number is subcritical ($Re < Re_c$). By means of a loudspeaker, a periodic excitation u'_x of the speed $u_{x\infty}$ of the upstream flow is produced on a frequency close to the natural one. An acoustic excitation was preferred to a direct mechanical vibration in order to reduce the noise; the streamwise excitation avoids three-dimensional effects though a transverse excitation would be more efficient since wakes are more unstable to antisymmetric disturbances. In the range of relevant frequencies, the relationship between the voltage applied to the loudspeaker and the energy of the excitation $u_x'^2$ has been determined, in the absence of any cylinder and at the same location in the wind tunnel (0, 0, 0). Different tests have shown that these curves are typical and vary neither with position nor with the speed in the wind tunnel. The diameters of the cylinders are 0.4 and 0.6 cm because their shedding frequencies are subharmonics of the maximum of efficiency of the loudspeaker (12 Hz). For every cylinder, the amplitude of the excitation is kept constant throughout the frequency range.

In the subcritical area we assume that we can neglect the nonlinear term in the Stuart–Landau equation (we will discuss this point further); therefore this equation becomes:

$$\frac{\mathrm{d}u_y}{\mathrm{d}t} = \sigma u_y + F, \tag{12}$$

The excitation term, $F = F_0(Re, \omega)\,\mathrm{e}^{\mathrm{i}\omega t}$, is characteristic of the hydrodynamic coupling between the excited upstream flow and the wake of the cylinder. In a similar study about the mixed-layer problem, Tam (1978) used an acoustic beam as an excitor.

A periodic solution of this equation (of the same frequency as the excitation) is

$$u_y = F_0(Re, \omega)\,\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{e}^{\mathrm{i}\psi}[\sigma_\mathrm{r}^2 + (\omega - \sigma_\mathrm{i})^2]^{-\frac{1}{2}} \tag{13}$$

and the amplitude

$$|u_y| = F_0(Re, \omega)\,[\sigma_\mathrm{r}^2 + (\omega - \sigma_\mathrm{i})^2]^{-\frac{1}{2}}. \tag{14}$$

The resonance is obtained for $\partial|u_y|/\partial\omega = 0$ when the angular frequency ω satisfies

$$\frac{\partial F_0}{\partial \omega}[\sigma_\mathrm{r}^2 + (\omega - \sigma_\mathrm{i})^2] - (\omega - \sigma_\mathrm{i})\,F_0 = 0. \tag{15}$$

In the experimental conditions, $Re \approx Re_\mathrm{c}$ so that $\sigma_\mathrm{r} \ll \sigma_\mathrm{i}$, and the angular frequency becomes

$$\omega = \sigma_\mathrm{i}\left[1 + \frac{(\sigma_\mathrm{r}^2/\sigma_\mathrm{i})(\partial F_0/\partial\omega)}{F_0(\sigma_\mathrm{i})}\right]. \tag{16}$$

Our hypothesis will be that, while keeping constant the longitudinal excitation u_x', the term $(\sigma_\mathrm{r}^2/\sigma_\mathrm{i})(\partial F_0/\partial\omega)/F_0(\sigma_\mathrm{i})$ is weak.

Thus there is a resonance characterized by:

a resonant angular frequency $\omega \approx \sigma_\mathrm{i}$,

an energy of the maximum $|u_y|^2 = \text{constant}/\sigma_\mathrm{r}^2 = \text{constant}/(Re - Re_\mathrm{c})^2$ (17)

a half height bandwidth frequency $\delta = 2|\sigma_\mathrm{r}| = 2\nu|Re - Re_\mathrm{c}|/5d^2$. (18)

The experiments show that the system, excited on the longitudinal component of the flow speed u_x, responds by shedding a street of alternately and oppositely circulating vortices of the same kind as that found for $Re > Re_\mathrm{c}$. The experimental resonance curves, response/excitation ratio versus frequency (figure 8), fit the theoretical pattern. These graphs point out two features:

the energy of the maximum depends on Re–Re_c

the bandwidth δ increases with $|Re - Re_\mathrm{c}|$.

The variation of the maximum in energy with the Reynolds number is plotted in

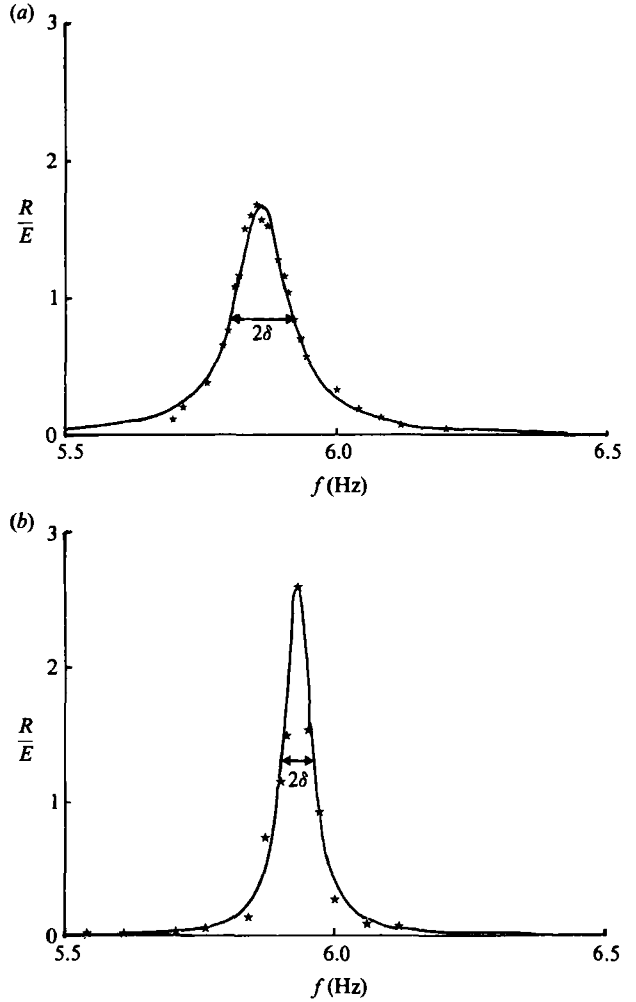

FIGURE 8. Resonances curves for a cylinder of diameter $d = 0.4$ cm. The response/excitation ratio is in arbitrary units because the response is on u_y while the excitation is on u_x. δ = half-height bandwidth frequency. (a) $Re - Re_c = -2.40$, (b) $Re - Re_c = -1.08$.

figure 9, this energy varies as the square of the difference to the critical Reynolds number (the accuracy of the determination of $Re - Re_c$ and the limitation in the exciting energy, due to the loudspeaker, made it impossible to obtain measurements on a range wider than two orders of magnitude).

Keeping the Reynolds number constant and increasing the energy of excitation to the limits of the loudspeaker, three orders of magnitude in energy, no significant change in the ratio response/excitation for the resonant maximum was observed. Moreover in all the experiments the value of $|u_y|^2$ is small compared to the energy of the free oscillation obtained for $Re > Re_c$ and the same value of $|Re - Re_c|$; thus the validity of our assumption concerning the nonlinear term is confirmed.

The dimensionless half height bandwidth frequency $\Delta Ro = \delta d^2/\nu$, plotted as a function of the Reynolds number, follows the theoretical relation (18) with an accuracy better than 5 % (figure 10).

Moreover we found (figure 11) that the dimensionless resonant frequency Ro follows

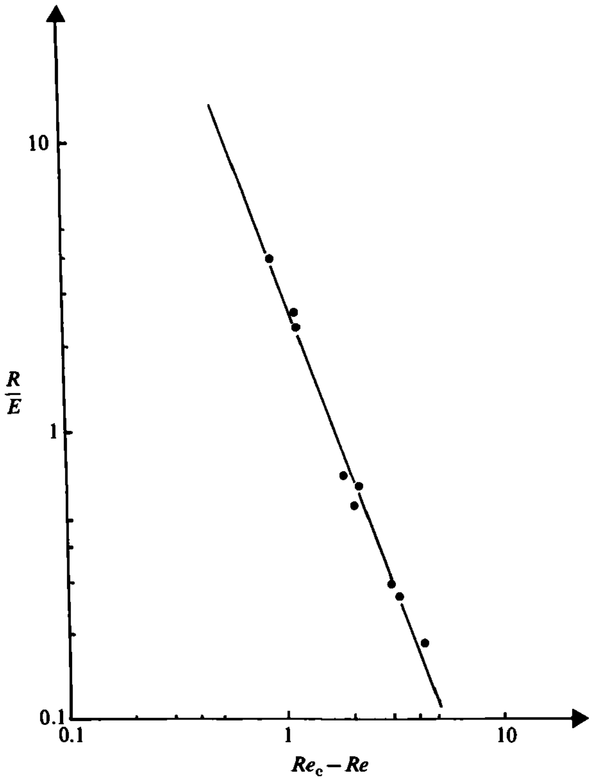

FIGURE 9. Maximum response energy/Excitation energy ratio versus $Re_c - Re$ in a log–log plot. $d = 0.40$ cm, $L/d = 25$.

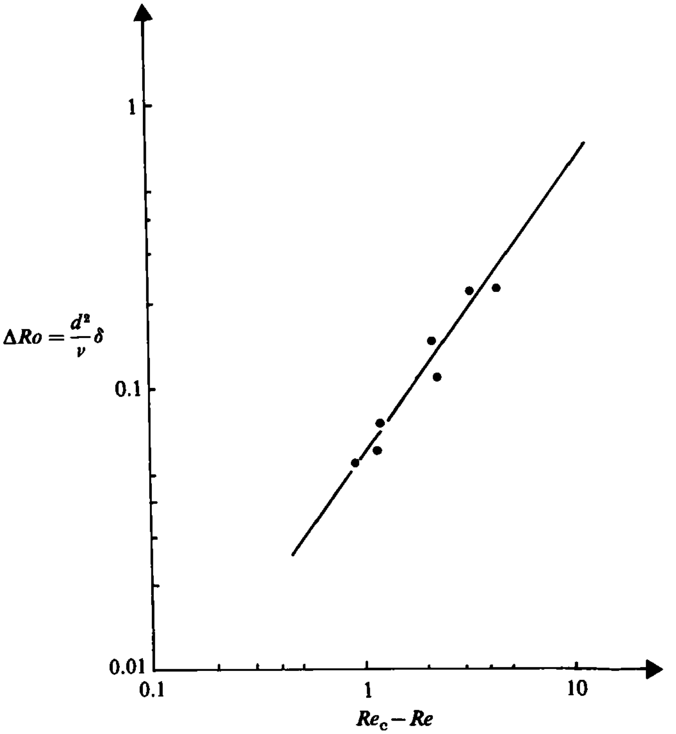

FIGURE 10. Dimensionless bandwidth frequency at half-height versus $Re_c - Re$ in a log–log plot. $d = 0.40$ cm.

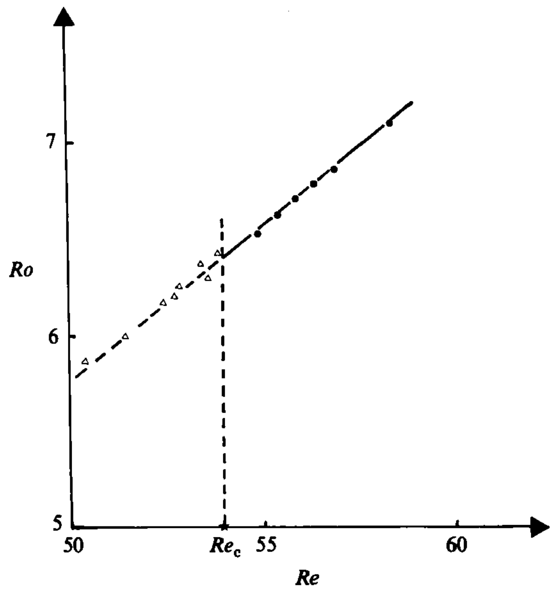

FIGURE 11. *Ro* versus *Re* beyond and above the critical point. $d = 0.40$ cm.

the same relation (equation (11)) as the supercritical. This result leads us to correct the interpretation of the Roshko–Reynolds relation proposed in Mathis *et al.* (1984*a*).

On one hand for $Re > Re_c$, the Stuart–Landau equation could be written:

$$\frac{du_y}{dt} = (\sigma_r + i\sigma_i)\, u_y - \tfrac{1}{2}(l_r + l_i)|u_y|^2 u_y \tag{5'}$$

which gives the two equations (2) and (6), for the amplitude,

$$\frac{d|u_y|}{dt} = \sigma_r |u_y| - \tfrac{1}{2} l_r |u_y|^3,$$

and for the phase,

$$\omega = \frac{d\phi}{dt} = \sigma_i - \tfrac{1}{2} l_i |u_y|^2.$$

On the other hand, in the subcritical area, the nonlinear term could be neglected and the Landau equation gives:

$$\frac{d|u_y|}{dt} = \sigma_r |u_y|$$

and

$$\omega = \sigma_i. \tag{19}$$

Near the critical point, the reduced Reynolds number $\epsilon = |Re - Re_c|/Re_c$ is small and we may develop σ_r and σ_i as two linear functions of $Re - Re_c$:

$$\sigma_r = k(Re - Re_c) + O(Re - Re_c)^2,$$

equation (3) where $k = \nu/5d^2$ and

$$\sigma_i = \sigma_{ic} + \gamma(Re - Re_c) + O(Re - Re_c)^2. \tag{20}$$

Therefore two expressions are obtained

(a) if $Re > Re_c$

$$|u_y|^2 = 2\sigma_r/l_r$$

$$\omega = \sigma_i - \tfrac{1}{2}l_i|u_y|^2 = \sigma_i - \sigma_r l_i/l_r = \sigma_{ic} + (\gamma - kl_i/l_r)(Re - Re_c) \qquad (6')$$

and it follows that

$$Ro = \frac{fd^2}{\nu} = \frac{\omega d^2}{2\pi\nu} = Ro_c + \frac{(\gamma - kl_i/l_r)(Re - Re_c)\,d^2}{2\pi\nu}, \qquad (21)$$

where $Ro_c = \sigma_{ic}\,d^2/2\pi\nu$.

(b) if $Re < Re_c$

$$\omega = \sigma_i = \sigma_{ic} + \gamma(Re - Re_c) + O(Re - Re_c)^2$$

so that

$$Ro = \frac{fd^2}{\nu} = \frac{\omega d^2}{2\pi\nu} = Ro_c + \frac{\gamma(Re - Re_c)\,d^2}{2\pi\nu}. \qquad (22)$$

At the critical point, the curves $Ro(Re)$ should exhibit two different slopes due to the variation of the shedding frequency $f_i = \sigma_i/2\pi$ in the subcritical area and due to both the influences of the frequency f_i and the nonlinear term $\frac{1}{2}l_i|u_y|^2$ in the supercritical area.

The experimental values (Re, Ro) near the critical point are plotted (figure 11) and the absence of any discontinuity in the slope of this graph suggests that the second term, $\beta l_i/l_r$, is negligible. This result validates Koch's assumption (1985) that nonlinear terms do not influence the shedding frequency of the wake.

4.2. *Forced oscillations in the supercritical regime*

Above the critical point, nonlinear oscillations forced by external excitation have been investigated. The literature devoted to this subject (Bergé, Pomeau & Vidal 1984; Kuramoto 1984) gives the different typical behaviours. While increasing the amplitude of excitation the spectrum of the signal exhibits (figure 12);

the natural frequency for $F_0 = 0$,
the two different frequencies f_i and f and their linear combination (by increasing the excitation the natural frequency f_i is shifted to the external one f),
for a critical value F_{0c} of the amplitude of the excitation, the free oscillation fades and the forced oscillator is synchronized to the external excitation.

These characteristics of nonlinear oscillations have been previously observed in the specific case of the wake cylinder (see survey article by Berger & Wille 1972). However if these researches, in the field of vortex-shedding excited oscillations, have attracted engineers' attention because of their considerable practical importance, they have been mostly conducted for high Reynolds numbers where problems of heat transfer and aeroelasticity occurred (Blevins 1985; Peterka & Richardson 1969; Toebes 1969). In our range of Reynolds numbers we noticed the works of Berger & Wille 1972, Hussain & Ramjee 1976 and Koopmann 1967. Unfortunately there is no systematic investigation of this problem and the data on the critical amplitude have a wide scatter.

Here we focus our attention on the conditions of excitation required by the synchronization. In the frame of a simplified Stuart–Landau model, typical results will be first presented; furthermore they will be extended to different experimental

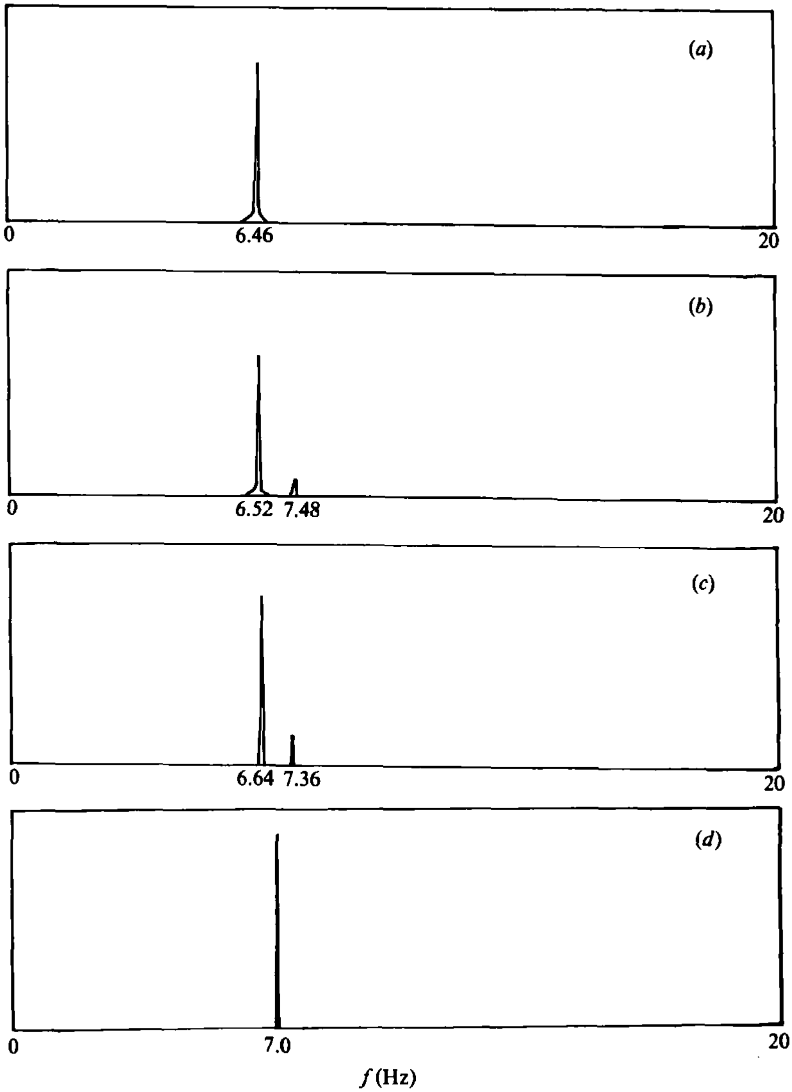

FIGURE 12. Typical spectra of the forced oscillations. (*a*) $F_0 = 0$, $f_i = 6.46$ Hz; (*b*) $F_0 = 0.5 F_{0c}$, $f = 14$ Hz, $f_i = 6.52$ Hz, $f - f_i = 7.48$ Hz; (*c*) $F_0 = 0.7 F_{0c}$, $f = 14$ Hz, $f_i = 6.64$ Hz, $f - f_i = 7.36$ Hz; (*d*) $F_0 \geqslant F_{0c}$, $f = 14$ Hz synchronization on 7 Hz. In (*b*) and (*c*) notice the change of the natural shedding frequency from the free value 6.46 Hz to the synchronized value 7.0 Hz.

conditions of resonance. The system is excited on a frequency close to the free one f_i; taking a simplified expression $F_0 \, e^{i\omega t}$ of the exciting term and assuming that l_i is equal to zero (§4.1), the Stuart–Landau equation can be written:

$$\frac{du_y}{dt} = (\sigma_r + i\sigma_i)\, u_y - \tfrac{1}{2}(l_r)\, |u_y|^2 u_y + F_0 \, e^{i\omega t}. \tag{23}$$

A solution $|u_y| \, e^{i\phi}$ of this equation verifies the following differential equations:

$$\frac{d|u_y|}{dt} = \sigma_r |u_y| - \tfrac{1}{2} l_r |u_y|^3 + F_0 \cos(\omega t - \phi), \tag{24}$$

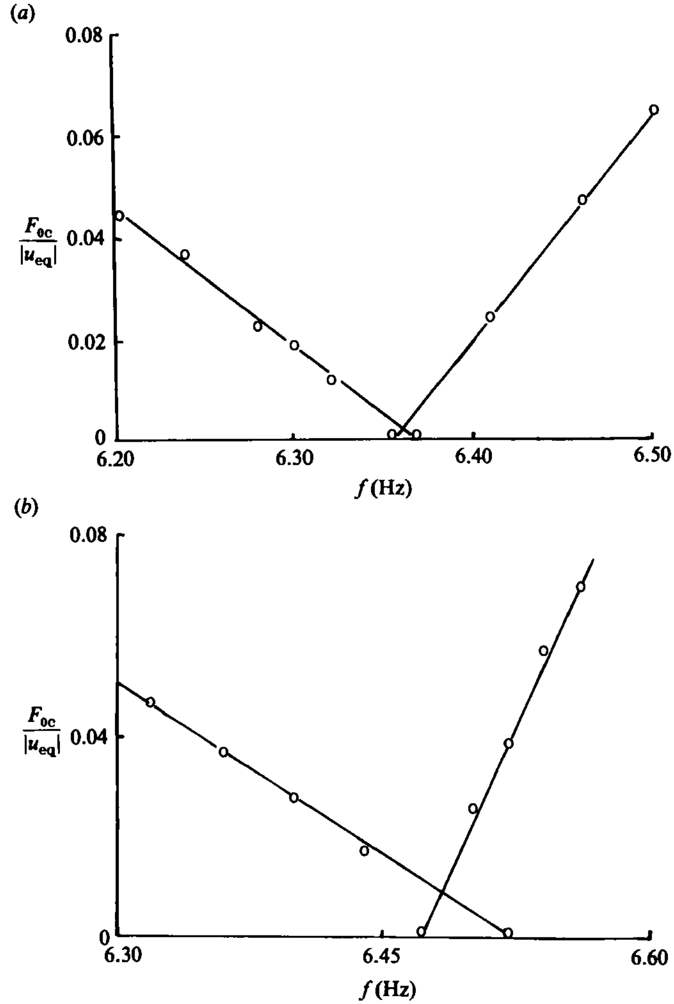

FIGURE 13. Critical forcing amplitudes *vs.* f for $f \approx f_1$. $d = 0.4$ cm, (a) $f_1 = 6.35, 6.37$ Hz. (b) $f_1 = 6.47$, 6.52 Hz. The different values of the natural frequency f_1 are due to a small variation of the upstream velocity induced by thermal effects.

$$\frac{|u_y|\,\mathrm{d}\phi}{\mathrm{d}t} = |u_y|\,\sigma_\mathrm{i} + F_0 \sin(\omega t - \phi). \tag{25}$$

Entrainment to the external periodicity occurs if these equations have a stable solution. Stationary synchronized solutions $|u_y| = \text{constant}$, $\phi = \omega t + \psi$, $\psi = \text{constant}$ are obtained in the following conditions:

$$\frac{d|u_y|}{\mathrm{d}t} = 0 = \sigma_\mathrm{r}|u_y| - \tfrac{1}{2}l_\mathrm{r}|u_y|^3 + F_0 \cos(\omega t - \phi),$$

$$|u_y|\,(\omega - \phi_1) = F_0 \sin\psi.$$

The phase condition gives the limit of synchronization:

$$|\sin\psi| = 1 \Rightarrow F_{0\mathrm{c}} = |u_y|\,|\omega - \sigma_1|. \tag{26}$$

As in the subcritical regime the system is excited by a loudspeaker.

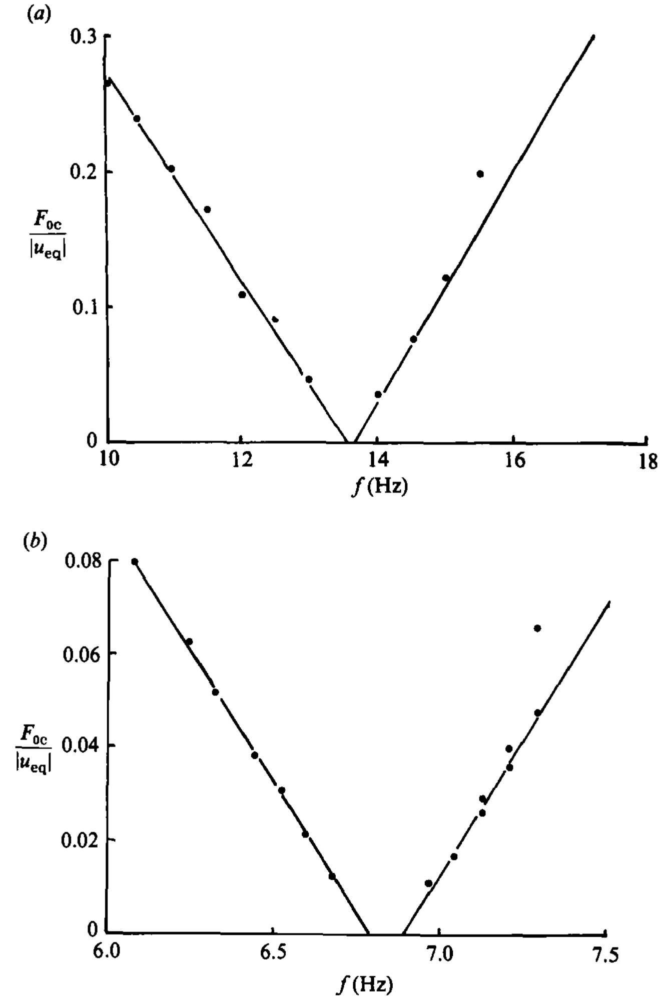

FIGURE 14. Critical forcing amplitudes *vs.* f for $f \approx 2f_1$. (*a*) $d = 0.4$ cm, (*b*) $d = 0.6$ cm. Notice the different points characteristic of hysteresis on the (*b*) plot.

For every frequency the critical level of entrainment F_{0c} has been determined; the measurements have been made on the point $(5d, 0, 0)$ where the amplitude of the natural oscillation is maximum and we have checked that the critical value does not change with the location.

First we verify that for weak amplitudes of the excitation ($F_0 < F_{0c}$) the entrainment is not realized when ω is different to the external one. In figure 13 the experimental points $F_{0c}/|u_{eq}|$, where $|u_{eq}|$ is the natural amplitude, are plotted versus the difference in frequency $(f - f_1)$ (in our conditions the energy of the entrained oscillation $|u_y|^2$ is the same as the free energy $|u_{eq}|^2$ so that the condition for entrainment (26) becomes $F_{0c}/|u_{eq}| = 2\pi|f - f_1|$). The linear relation is well verified, but we cannot check the numerical values because the forcing F_0 is deduced from a calibration on $|u_x|$, while the response is measured on $|u_y|$. Similar curves (but not V shaped) have been plotted by Koopmann 1967, in the case of a vibrating cylinder, on a large frequency range ($\Delta f/f = \pm 25\,\%$ around f_1); Koopmann reported a threshold value of the critical amplitude which we do not observe.

The experimental curves are not perfectly symmetric; the entrainment seems more difficult to obtain for high values of the forcing frequency (figure 13*b*). Such a behaviour has been observed before by Blevins (1985) ($Re \approx 2\times10^4$) who reported that 'the shift to the applied frequency is greater when the sound is applied at frequencies below the shedding frequency'. Moreover for large differences $|f-f_1|$, two critical amplitudes F_{0c^+}, F_{0c^-} appear in the experiments; the first one occurs when increasing the excitation in order to get entrainment, the second characterizes the desynchronization while decreasing the forcing. This phenomenon has been first observed by Bishop & Hassan (1964). Such a hysteresis is typical of nonlinear oscillations. Obviously in this case the simplified model is not sufficient and additional terms must be taken into account in the Stuart–Landau equation. For small values of $|f-f_1|$ this phenomenon probably exists but the uncertainty of measurements do not allow one to appreciate the different values of entrainment. Limitation of the energy of the generator oblige one to work near the critical Reynolds number, where the amplitude of the free oscillation is small; thus experimental constraints reduce the study of the entrainment to a narrow range of frequency $\Delta f = 0.4$ Hz or $\Delta f/f = 8\,\%$.

The investigation was carried on by exciting the system on a frequency close to the first harmonic of the natural one. The behaviour of this system is the same as that on the simple frequency. The spectrum of the signal shows the forcing frequency and the free shift frequency for weak forcing levels. The entrainment of the oscillation on the half frequency of the excitation appears above a critical value of the forcing level F_{0c}; this frequency demultiplication, also called frequency division, is classic in experiments on oscillators and has been observed by Hussain & Ramjee (1976) (see Berger & Wille (1972) for references in different Reynolds number range). Plots of the curves $(f-2f_1, F_{0c}/|u_{eq}|)$ present the same V shape (figure 14); for large values of $f-2f_i$, hysteresis appears in the determination of the critical levels F_{0c}. As for the simple frequency the entrainment is more difficult to get for high than for low frequency.

Different tests on sub- or super-harmonics frequency did not allow us to observe the synchronization with the present experimental device. Further experiments on a stronger excitator are needed to determine the role taken by the double frequency. However we note that nonlinear terms of the Stuart–Landau equation change with the frequency of the external forcing: the expression of the resonant term in the development of the normal form changes with the ratio of the natural frequency to the frequency of the excitation (Gambaudo 1983).

5. Conclusion

The results of this present work confirm our first observation on the temporal behaviour of the Bénard–von Kármán instability near the threshold. To our knowledge this is one of the first studies of an external flow considered as a dynamic system. The Stuart–Landau equation is well adapted to describe the behaviour not only beyond the threshold but also below, when the system is submitted to forced excitation. The coefficients of the Stuart–Landau model have been determined and it must be noticed that the imaginary part l_i of the nonlinear term $|u_y|^2u_y$ is negligible. This result agrees with the usual assumption. Thus nonlinearity is essential to determine the appropriate amplitudes of the instability but is of negligible influence upon the frequency of the oscillation which is a characteristic of the wake behind a bluff body. Moreover we have observed the existence of synchronization on an external forcing. The stable zone (amplitude, frequency) exhibits the same **v** shape as a Van der Pol oscillator whose equation can be reduced to a Landau model. This property exists for a frequency equal to, as well as double, the free oscillation.

This study of temporal behaviour can be considered as a first step towards a study of three-dimensional effects. In fact the Stuart–Landau equation can be extended to a Ginzburg–Landau equation by addition of a diffusive term. From an experimental point of view this field of nonlinear oscillators could be investigated by a study of the phase dependence as a function of z along the rod axis for instance. However a numerical study of the basic wake profile would be of interest for computing the phenomenological coefficients of the Stuart–Landau equation from the actual problem of the wake of a cylinder. Such a study would also permit one to clarify the coupling between the excited upstream flow and the self excited vortex-shedding process.

To conclude, it may be worthwhile pointing out why this model is interesting. It has been seen that such a phenomenological approach does not describe the vortex shedding mechanism any more than the exact Navier–Stokes equations do. But by showing certain characteristic properties it allows us to put the system in a class of oscillators which share its properties and by analogy to suggest specific behaviour and some embryo of an explanation.

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Appendix

a, b	constants in the non-dimensional frequency–velocity relation
d	cylinder diameter
f	frequency
f_i	natural shedding frequency
F	excitation induced by the loudspeaker
$F_0(Re, \omega)$	amplitude of the excitation
F_{0c}	critical amplitude for which the shedding of the wake is synchronized
F_{0c^+}	critical amplitude when increasing the forcing
F_{0c^-}	critical amplitude when decreasing the forcing
k	characteristic diffusion frequency
l	coefficient of the nonlinear term in the Stuart–Landau equation
l_i	imaginary part of l
l_r	real part of l
L	cylinder length
$\frac{R}{E}$	response energy/excitation energy ratio $= \frac{\lvert u_y\rvert^2}{\lvert u'_x\rvert^2}$
Re	Reynolds number $= U_{x\infty}\, d/\nu$
Re_c	critical Reynolds number, Re_c varies with the aspect ratio L/d of every cylinder
Re_i	initial Reynolds number
Re_f	final Reynolds number
Ro	Roshko number = dimensionless frequency $= fd^2/\nu$
Ro_c	Roshko number's value at the critical point ($Re = Re_c$)
T	period of the oscillation
$\lvert u_{eq}\rvert$	amplitude of the natural shedding transverse velocity fluctuation

u_x, u_y, u_z streamwise, transverse and spanwise velocity fluctuations
u'_x streamwise velocity fluctuation induced by the loudspeaker
U_x, U_y, U_z streamwise, transverse and spanwise mean velocities
$U_{x\infty}$ free-stream mean velocity
x, y, z streamwise, transverse and spanwise coordinates
α cofactor in the amplitude of the shedding oscillation $|u_y|$
γ coefficient of the angular frequency–Reynolds number development
δ half-height bandwidth frequency
ΔRo dimensionless half-height frequency $= \delta d^2/\nu$
ϵ reduced Reynolds number $= |Re - Re_c|/Re_c$
ϕ phase of u_y
ν kinematic viscosity
σ coefficient of the linear term in the Stuart–Landau equation
σ_i imaginary part of σ = natural angular frequency
σ_{ic} critical angular frequency
σ_r real part of σ
τ transient time evolution
ψ response–excitation phase shift when the response is synchronized
ω angular frequency

REFERENCES

Bénard, H. 1908 Formation des centres de giration à l'arrière d'un obstacle en mouvement. *C. R. Acad. Sci. Paris* **147**, 839.

Bergé, P., Pomeau, Y. & Vidal, C. 1984 *L'ordre dans le Chaos.* Hermann.

Berger, E. & Wille, R. 1972 Periodic flow phenomena. *Ann. Rev. Fluid Mech.* **4**, 313–340.

Bishop, R. & Hassan, A. 1964 The lift and drag forces on a circular cylinder in a flowing fluid. *Proc. R. Soc. Lond.* A **277**, 32–50.

Blevins, R. D. 1985 The effect of sound on vortex shedding from cylinders. *J. Fluid Mech.* **161**, 217–237.

Braza, M. 1981 Simulation numérique du décollement instationnaire externe par une formulation vitesse-pression. Application à l'écoulement autour d'un cylindre. Thesis, Université de Toulouse.

Coutanceau, M. & Bouard, R. 1977 Experimental determination of the main features of the viscous flow in the wake of a circular cylinder in uniform translation. *J. Fluid Mech.* **79**, 237.

Drazin, P. G. & Reid, W. H. 1981 *Hydrodynamic Stability.* Cambridge University Press.

Fornberg, B. 1980 A numerical study of steady viscous flow past a circular cylinder. *J. Fluid Mech.* **98**, 819–855.

Gambaudo, J. M. 1983 Perturbation of a Hopf bifurcation by an external time-periodic forcing. Thesis, Université de Nice.

Gaster, M. 1971 Vortex shedding from circular cylinders at low Reynolds numbers. *J. Fluid Mech.* **46**, 565–576.

Gollub, J. & Freilich, M. 1981 Critical exponents and generalized potential for the Taylor instability. In *Fluctuations, Instabilities and Phase Transitions* (ed. T. Riste), pp. 195–203. NATO Report.

Haldenwang, P. & Labrosse, G. 1986 2D and 3D spectral Chebyshev solutions for free convection at high Rayleigh number. *Proc. Sixth Intl Symp. on Finite element methods in flow problems*, Antibes, pp. 261–267.

Hussain, A. K. M. F. & Ramjee, V. 1976 Periodic wake behind a circular cylinder at low Reynolds numbers. *Aero. Q.* May 123–141.

Jackson, C. P. 1987 A finite element study of the onset of vortex-shedding past variously shaped bodies. *J. Fluid Mech.* **182**, 23–45.

JOSEPH, D. D. 1981 Hydrodynamic stability and bifurcation. In *Hydrodynamics Instabilities and the Transition to Turbulence* (ed. H. Swinney & J. Gollub), pp. 27–76. Springer.

KÁRMÁN, T. VON 1911 Uber den Mechanismus der Widerstander den ein bewegter Körper in einer Flüsseigkeit afahrt. *Gott. Nachr*, 509–517.

KOCH, W. 1985 Local instability characteristics and frequency determination of self excited wake flows. *J. Sound Vib.* **99**, 53–83.

KOOPMANN, G. H. 1967 The vortex wakes of vibrating cylinders at low Reynolds numbers. *J. Fluid Mech.* **28**, 501–512.

KURAMOTO, Y. 1984 *Chemical Oscillations, Waves and Turbulence.* Springer.

LANDAU, L. D. 1944 On the problem of turbulence. *C.R. Acad. Sci. URSS* **44**, 311–314.

LANDAU, L. & LIFSHITZ, E. 1971 *Mécanique des Fluides.* Moscow: Mir.

LIBCHABER, A. & MAURER, J. 1981 A Rayleigh–Bénard experiment: helium in a small box. In *Nonlinear Phenomena at Phase Transitions and Instabilities* (ed. T. Riste), pp. 259–286. NATO Report.

MACLAUGHLIN, J. B. & ORSZAG, S. 1982 Transition from periodic to chaotic thermal convection. *J. Fluid Mech.* **122**, 123–142.

MATHIS, C. 1983 Propriétés des composantes de vitesse transverses dans l'écoulement de Bénard–von Kármán aux faibles nombres de Reynolds. Thesis, Université Aix–Marseille.

MATHIS, C., PROVANSAL, M. & BOYER, L. 1984*a* The Bénard–von Kármán instability: an experimental study near the threshold. *J. Phys. Paris Lett.* **45**, 483–491.

MATHIS, C., PROVANSAL, M. & BOYER, L. 1984*b* The rotating grating applied to the study of the Bénard–von Kármán instability near the threshold. *Proc. Second Intl Symp. on Applications of Laser Anemometry to Fluid Mechanics, Lisbon*, p. 12.5.

MEIRON, D., SAFFMAN, P. G. & SCHATZMAN, J. 1984 The linear two-dimensional stability of inviscid vortex streets of finite-cored vortices. *J. Fluid Mech.* **147**, 187–212.

NAKAMURA, I. 1976 Steady wake behind a sphere. *Phys. Fluids*, **19**, 5–8.

PALM, E. 1960 On the tendency towards hexagonal cells in steady convection. *J. Fluid Mech.* **8**, 183–192.

PETERKA, J. & RICHARDSON, P. 1969 Effect of sound on separated flows. *J. Fluid Mech.* **37**, 265–287.

ROSHKO, A. 1954 On the development of turbulent wakes from vortex streets. *NACA Tech. Note* 2913, 1–71.

RUELLE, D. & TAKENS, F. 1979 On the nature of turbulence. *Commun. Math. Phys.* **20**, 167–192.

SAFFMAN, P. G. & SCHATZMAN, J. C. 1982 Stability of vortex street of finite vortices. *J. Fluid Mech.* **117**, 171–185.

SREENIVASAN, K. R. 1985 Transition and turbulence in fluid flows and low-dimensional chaos. In *Frontiers in Fluid Mechanics* (ed. S. H. Davis & J. L. Lumley), pp. 41–66. Springer.

STRYKOWSKI, P. J. 1986 The control of absolutely and convectively unstable flows. PhD thesis, Yale University.

STUART, J. T. 1958 On the non-linear mechanics of hydrodynamic stability. *J. Fluid Mech.* **4**, 1–21.

STUART, J. T. 1960 On the non-linear mechanics of wave disturbances in stable and unstable parallel flows. *J. Fluid Mech.* **9**, 353–370.

TAM, C. 1978 A note on vortex streets behind circular cylinders at low Reynolds numbers. *J. Fluid Mech.* **45**, 203–208.

TOEBES, G. 1969 The unsteady flow and wake near an oscillating cylinder. *Trans ASME* D: *J. Basic Engng* **91**, 493–505.

TRITTON, D. 1971 A note on vortex streets behind circular cylinders at low Reynolds numbers. *J. Fluid Mech.* **45**, 203–208.